CONVOLUTION EQUATIONS IN SPACES OF
DISTRIBUTIONS WITH ONE-SIDED BOUNDED SUPPORT

BY

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ABSTRACT. Let $\mathscr{D}'(0, \infty)$ be the space of distributions on R with support in $[0, \infty)$ and $\mathscr{S}'(0, \infty)$ its subspace consisting of tempered distributions. We characterize the distributions $S \in \mathscr{D}'(0, \infty)$ for which $S * \mathscr{D}'(0, \infty) = \mathscr{D}'(0, \infty)$, where $*$ is the convolution. We also characterize the distributions $S \in \mathscr{S}'(0, \infty)$ for which $S * \mathscr{S}'(0, \infty) = \mathscr{S}'(0, \infty)$.

The problem of existence of solutions of convolution equations in spaces of distributions on R^n was thoroughly investigated by L. Ehrenpreis [1]. Later, L. Hörmander [2] gave a systematic exposition of the range of convolution operators in spaces of distributions on arbitrary open sets $\Omega \subset R^n$. In particular, consider the space $\mathscr{D}'$ of all distributions on R^n and its subspace $\mathscr{E}'$ consisting of distributions of compact support. If $S \in \mathscr{E}'$, the convolution equation

$$(1) \qquad\qquad S * u = v$$

has a solution $u \in \mathscr{D}'$, for every $v \in \mathscr{D}'$, if and only if the Fourier transform $\hat{S}$ of S has the following property:

There are constants A_1, A_2 and A_3 such that for every $\xi \in R^n$ there exists $\eta \in R^n$ satisfying the conditions

$$(2) \qquad |\xi - \eta| \leqslant A_1 \log(2 + |\xi|) \quad \text{and} \quad |\hat{S}(\eta)| \geqslant (A_2 + |\xi|)^{-A_3}.$$

In that case S is said to be invertible and $\hat{S}$ slowly decreasing.

However, if we consider the space $\mathscr{D}'(0, \infty)$ of distributions of one variable with support in $[0, \infty)$, then the convolution $S * u$ is defined for any $S, u \in \mathscr{D}'(0, \infty)$ as a distribution in $\mathscr{D}'(0, \infty)$. The question now arises: Under what conditions on $S \in \mathscr{D}'(0, \infty)$ has equation (1) a solution $u \in \mathscr{D}'(0, \infty)$ for every $v \in \mathscr{D}'(0, \infty)$? Since the Fourier transform $\hat{S}$ of a distribution $\hat{S} \in \mathscr{D}'(0, \infty)$ is, in general, not even a distribution, conditions (2) are now meaningless.

In this paper we characterize the distributions $S \in \mathscr{D}'(0, \infty)$ for which $S * \mathscr{D}'(0, \infty) = \mathscr{D}'(0, \infty)$. We also solve the problem of existence of solutions of equation (1) in the space $\mathscr{S}'(0, \infty)$ of tempered distributions with support in $[0, \infty)$.

1. Preliminaries. We denote by $\mathscr{D}_-$ the space of all C^∞-functions with support bounded from the right and by $\mathscr{D}'_+$ its dual space. $\mathscr{D}'_+$ is the space of distributions with support bounded from the left. We also denote by $\mathscr{D}'(\alpha, \infty)$ the space of distributions with support in $[\alpha, \infty)$. For details about these spaces we refer to [4].

If $S, T \in \mathscr{D}'_+$, then the convolution $S * T$ is well defined as a distribution in $\mathscr{D}'_+$. We recall some of the properties of the convolution which we apply later. First we have

$$(3) \qquad\qquad \operatorname{supp}(S * T) \subset \operatorname{supp} S + \operatorname{supp} T.$$

In particular, if $S \in \mathscr{D}'(\alpha, \infty)$ and $T \in \mathscr{D}'(\beta, \infty)$, then $S * T \in \mathscr{D}'(\alpha + \beta, \infty)$.

Furthermore, the following theorem of Titchmarsh (see [3 and 6]) is valid for distributions in $\mathscr{D}'(0, \infty)$:

If S and T are distributions in $\mathscr{D}'(0, \infty)$ and $S * T \in \mathscr{D}'(\gamma, \infty)$, $\gamma > 0$, then $S \in \mathscr{D}'(\alpha, \infty)$ and $T \in \mathscr{D}'(\beta, \infty)$, where $\alpha + \beta = \gamma$, $\alpha, \beta \geqslant 0$.

Finally, if $S \in \mathscr{E}'$, then the value of the convolution $S * T$ on an open set Ω depends only on the value of T in $\Omega - \operatorname{supp} S$.

We denote by $\mathscr{S}'(0, \infty)$ the space of tempered distributions with support in $[0, \infty)$. Distributions in $\mathscr{S}'(0, \infty)$ can be characterized in terms of their Fourier transforms (see [5]). Let $C_- = \{\zeta \in C: \operatorname{Im} \zeta < 0\}$ and denote by M_n the function defined on $(-\infty, 0)$ by

$$(4) \qquad\qquad M_n(\eta) = \begin{cases} 1 & \text{if } \eta \leqslant -1, \\ |\eta|^{-n} & \text{if } -1 < \eta < 0; \end{cases}$$

here n is an integer. A tempered distribution T is in $\mathscr{S}'(0, \infty)$ if and only if its Fourier transform $\hat{T}$ can be continued in C_- to an analytic function $\hat{t}$ satisfying the estimate

$$(5) \qquad\qquad |\hat{t}(\zeta)| \leqslant C(1 + |\zeta|)^m M_n(\eta),$$

where $\zeta = \xi + i\eta$, m and n are positive integers and C is a constant.

If S and T are both in $\mathscr{S}'(0, \infty)$ and if $\hat{s}$ and $\hat{t}$ are the corresponding analytic continuations in C_- of their Fourier transforms, then $\hat{s}\hat{t}$ is the analytic continuation of a tempered distribution $\hat{S} \circ \hat{T}$ such that

$$\widehat{S * T} = \hat{S} \circ \hat{T}.$$

2. The case of convolution operators of compact support. We first consider convolution operators S with compact support and assume that $\operatorname{supp} S \subset [0, \infty)$; in that case $S * \mathscr{D}'(0, \infty) \subset \mathscr{D}'(0, \infty)$.

If $S * \mathscr{D}'(0, \infty) = \mathscr{D}'(0, \infty)$, then $0 \in \operatorname{sing} \operatorname{supp} S$. In fact, if $\operatorname{sing} \operatorname{supp} S \subset [\alpha, \infty)$, $\alpha > 0$, then $\operatorname{sing} \operatorname{supp} S * u \subset [\alpha, \infty)$ for every $u \in \mathscr{D}'(0, \infty)$, since the value of $S * u$ in $(-\infty, \alpha)$ depends only on the value of S in $(-\infty, \alpha)$. This shows that $S * \mathscr{D}'(0, \infty) \neq \mathscr{D}'(0, \infty)$.

In what follows we use the notion of invertibility of a convolution operator $S \in \mathscr{E}'$ in the same sense as in [1] (see the introduction). We also denote by R_j^- and R_j^+, $j = 0, \pm 1, \pm 2, \ldots$, the intervals $(-\infty, j]$ and $[j, \infty)$, respectively.

THEOREM 1. *If $S * \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$, then S is invertible and satisfies the conditions*

(6)
$$u \in \mathcal{D}'_+, \quad \text{sing supp}(S * u) \subset R_j^+ \Rightarrow \text{sing supp } u \subset R_j^+, \quad j = 0, \pm 1, \pm 2, \ldots.$$

PROOF. If $\varphi \in \mathcal{D}$, we pick $h \geqslant 0$ such that $\text{supp } \tau_h \varphi \subset [0, \infty)$. By assumption, there exists $u \in \mathcal{D}'(0, \infty)$ such that $S * u = \tau_h \varphi$. Hence $S * \tau_{-h} u = \varphi$, and, obviously, $\tau_{-h} u \in \mathcal{D}'$. In other words, we proved that $S * \mathcal{D}' \supset \mathcal{D}$, which implies that S is invertible by Theorem 2.5 in [1].

Furthermore, it follows from the hypothesis that S has a fundamental solution in $\mathcal{D}'(0, \infty)$, i.e. there exists $E \in \mathcal{D}'(0, \infty)$ such that $S * E = \delta$, where δ is the Dirac measure. Consequently, every distribution $u \in \mathcal{D}'(0, \infty)$ can be represented in the form $u = (S * u) * E$. If now $\text{sing supp}(S * u) \subset R_j^+$, then $S * u$ is a C^∞-function on the complement of R_j^+, and therefore $u = (S * u) * E$ is a C^∞-function on $(-\infty, j)$. Thus $\text{sing supp } u \subset R_j^+$.

REMARK 1. If S is any distribution in $\mathcal{D}'(0, \infty)$ such that $0 \in \text{supp } S$, then it follows from the Titchmarsh theorem that

(7)
$$u \in \mathcal{D}'_+, \quad \text{supp}(S * u) \subset R_j^+ \Rightarrow \text{supp } u \subset R_j^+,$$

$j = 0, \pm 1, \pm 2, \ldots.$ In fact, pick an integer $k \geqslant 0$ such that $\tau_k u \in \mathcal{D}'(0, \infty)$. If $\text{supp}(S * u) \subset R_j^+$, we have $\text{supp}(S * \tau_k u) = \text{supp } \tau_k(S * u) \subset R_{j+k}^+$. But $0 \in \text{supp } S$, and so $\text{supp } \tau_k u \subset R_{j+k}^+$, by Titchmarsh's theorem. Hence $\text{supp } u \subset R_j^+$.

REMARK 2. Conditions (6) and (7) are equivalent to the conditions

(8)
$$u \in \mathcal{D}'_-, \quad \text{sing supp}(\check{S} * u) \subset R_j^- \Rightarrow \text{sing supp } u \subset R_j^-,$$

(9)
$$u \in \mathcal{D}'_-, \quad \text{supp}(\check{S} * u) \subset R_j^- \Rightarrow \text{supp } u \subset R_j^-,$$

$j = 0, \pm 1, \pm 2, \ldots,$ where $\check{S}$ is the symmetric distribution to S. This follows immediately from the relation $(\check{S} * u)^\vee = S * \check{u}$.

We now prove the converse of Theorem 1.

THEOREM 2. *If S is invertible and satisfies conditions (6), then $S * \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$.*

PROOF. It suffices to show that

(10)
$$S * \mathcal{D}'_+ = \mathcal{D}'_+$$

and then apply Titchmarsh's theorem. The proof of (10) is basically the same as that of Theorem 4.5 in [2]. We therefore restrict ourselves to a brief sketch of the proof.

Let v be an arbitrary distribution in $\mathcal{D}'_+$ and let p be a seminorm in $\mathcal{D}_-$ such that

(11)
$$|v(\varphi)| \leqslant p(\varphi), \quad \varphi \in \mathcal{D}_-.$$

The main part of the proof is the construction of a seminorm q in $\mathcal{D}_-$ satisfying the condition

(12)
$$p(\varphi) \leqslant q(\check{S} * \varphi), \quad \varphi \in \mathcal{D}_-.$$

If this is accomplished, conditions (11) and (12) yield $|v(\varphi)| \leqslant q(\check{S} * \varphi)$, $\varphi \in \mathscr{D}_-$, and hence, by the Hahn-Banach theorem, the linear form $\check{S} * \varphi \to v(\varphi)$, $\varphi \in \mathscr{D}_-$, can be extended to a linear form u on $\mathscr{D}_-$ such that $|u(\psi)| \leqslant q(\psi)$, $\psi \in \mathscr{D}_-$. Thus $u \in \mathscr{D}'_+$, and since $u(\check{S} * \varphi) = v(\varphi)$, $\varphi \in \mathscr{D}_-$, we have $S * u(\varphi) = v(\varphi)$, $\varphi \in \mathscr{D}_-$, or $S * u = v$.

The construction of q is based on the following lemma.

LEMMA. *Let q be a seminorm in $\mathscr{D}_-$ such that*

$$q(\psi) \geqslant \sup_{\alpha \leqslant x} |\psi(x)|, \qquad \psi \in \mathscr{D}_-,$$

for some $\alpha \in R$, and assume that

$$p(\varphi) \leqslant q(\check{S} * \varphi), \qquad \varphi \in \mathscr{D}_-, \operatorname{supp} \varphi \subset R_j^-,$$

where $j > \alpha$. Then, under the conditions of the theorem, for every $\varepsilon > 0$ there exists another seminorm q' in $\mathscr{D}_-$ such that

$$p(\varphi) \leqslant q'(\check{S} * \varphi), \qquad \varphi \in \mathscr{D}_-, \operatorname{supp} \varphi \subset R_{j+1}^-,$$

and

$$q'(\psi) = (1 + \varepsilon)q(\psi), \qquad \psi \in \mathscr{D}_-, \operatorname{supp} \psi \subset R_{j-1}^-.$$

We observe that $q(\psi)$ depends only on the values of ψ in R_λ^+ for some $\lambda \in R$. Otherwise one could find a sequence of functions $\psi_n \in \mathscr{D}_-$ such that $\operatorname{supp} \psi_n \subset R_{\lambda_n}^-$, where $\lambda_n \to -\infty$, and $q(\psi_n) \geqslant 1$. Since $\psi_n \to 0$ in $\mathscr{D}_-$, this would be a contradiction.

In view of that, the conditions of the lemma need only to be satisfied by functions with compact support and the proof is then the same as in [2].

Given $\varepsilon_j > 0$ such that $\sum_{j=1}^\infty \varepsilon_j < \infty$, one can now apply the lemma to successively construct seminorms q_j in $\mathscr{D}_-$ such that

$$q_{j+1}(\psi) = (1 + \varepsilon_j)q_j(\psi), \qquad \psi \in \mathscr{D}_-, \operatorname{supp} \psi \subset R_{j-1}^-,$$

and

(13) $p(\varphi) \leqslant q_j(\check{S} * \varphi), \qquad \varphi \in \mathscr{D}_-, \operatorname{supp} \varphi \subset R_{j+1}^-.$

If $q(\psi) = \lim_{j \to \infty} q_j(\psi)$, then q is a continuous seminorm in $\mathscr{D}_-$ and conditions (13) imply (12).

COROLLARY 1. *For $S \in \mathscr{E}'$ with $\operatorname{supp} S \subset [0, \infty)$, the following conditions are equivalent.*

(i_1) $S * \mathscr{D}'(0, \infty) = \mathscr{D}'(0, \infty)$.

(i_2) $S * \mathscr{D}'_+ = \mathscr{D}'_+$.

(i_3) There exists a fundamental solution for S in $\mathscr{D}'(0, \infty)$.

(i_4) S is invertible and satisfies conditions (6).

3. The general case. We now consider equation (1), where S is an arbitrary distribution in $\mathscr{D}'(0, \infty)$. We prove that the solvability of this equation in $\mathscr{D}'(0, \infty)$ depends only on the values of S in an arbitrary small neighborhood of the origin.

THEOREM 3. *If* $S \in \mathscr{D}'(0, \infty)$ *and* $S * \mathscr{D}'(0, \infty) = \mathscr{D}'(0, \infty)$, *then, for every* $\alpha > 0$, *S admits a decomposition*

$$(14) \qquad\qquad\qquad\qquad S = S_1 + S_2,$$

where S_1 *is a distribution in* $\mathscr{E}'$ *satisfying the equivalent conditions in Corollary* 1, *and* $S_2 \in \mathscr{D}'(\alpha, \infty)$.

PROOF. Given $\alpha > 0$, we can always decompose S as in (14), with $S_1 \in \mathscr{E}'$ and $S_2 \in \mathscr{D}'(\alpha, \infty)$. In fact, if ψ is a function in $\mathscr{D}$ such that $\psi = 1$ on $[-\alpha, \alpha]$, we can define $S_1 = \psi S$ and $S_2 = S - S_1$.

By the hypothesis, there exists $E \in \mathscr{D}'(0, \infty)$ such that $S * E = \delta$. Hence

$$(15) \qquad\qquad\qquad\qquad S_1 * E = \delta - S_2 * E$$

and $S_2 * E \in \mathscr{D}'(\alpha, \infty)$, by (3).

Consider now a solution $u_1 \in \mathscr{D}'(0, \infty)$ of the equation $S * u = S_2 * E$. By assumption, u_1 exists. Also, since $S_2 * E \in \mathscr{D}'(\alpha, \infty)$ and $0 \in \operatorname{sing\,supp} S$, u_1 is in $\mathscr{D}'(\alpha, \infty)$ by Titchmarsh's theorem. It follows that

$$(16) \qquad\qquad\qquad\qquad S_1 * u_1 = S_2 * E - S_2 * u_1$$

and $S_2 * u_1 \in \mathscr{D}'(2\alpha, \infty)$ by (3).

Suppose that $u_k \in \mathscr{D}'(k\alpha, \infty)$ has been defined for $k \geqslant 1$. We then define u_{k+1} as a solution in $\mathscr{D}'(0, \infty)$ of the equation $S * u = S_2 * u_k$. Since $S_2 * u_k \in \mathscr{D}'((k+1)\alpha, \infty)$, u_{k+1} must be in $\mathscr{D}'((k+1)\alpha, \infty)$ by Titchmarsh's theorem, and we have

$$(17) \qquad\qquad\qquad\qquad S_1 * u_{k+1} = S_2 * u_k - S_2 * u_{k+1}.$$

In this way we have defined a sequence of distributions $u_k \in \mathscr{D}'(k\alpha, \infty)$ satisfying equations (16) and (17). Adding both sides of equations (15), (16), and (17) for $k = 1, 2, \ldots, n$, we obtain

$$(18) \qquad\qquad\qquad S_1 * \left(E + \sum_{k=1}^{n+1} u_k \right) = \delta - S_2 * u_{n+1}.$$

Since $u_n \in \mathscr{D}'(n\alpha, \infty)$, the series $\sum_{k=1}^{\infty} u_k$ converges and $S_2 * u_{n+1} \to 0$ in $\mathscr{D}'(0, \infty)$, as $n \to \infty$. Thus, from (18) it follows that

$$S_1 * \left(E + \sum_{k=1}^{\infty} u_k \right) = \delta,$$

i.e. $E_1 = E + \sum_{k=1}^{\infty} u_k$ is a fundamental solution for S_1 in $\mathscr{D}'(0, \infty)$.

The converse of Theorem 3 is also true.

THEOREM 4. *Let* S *be a distribution in* $\mathscr{D}'(0, \infty)$ *and let* $\alpha > 0$. *If* S *admits a decomposition* (14), *where* $S_1 \in \mathscr{E}'$, $S_2 \in \mathscr{D}'(\alpha, \infty)$, *and* S_1 *satisfies the equivalent conditions of Corollary* 1, *then* $S * \mathscr{D}'(0, \infty) = \mathscr{D}'(0, \infty)$.

PROOF. It suffices to construct a fundamental solution $E \in \mathscr{D}'(0, \infty)$ for S.

By assumption, there exists $E_1 \in \mathscr{D}'(0, \infty)$ such that $S_1 * E_1 = \delta$. We now consider a solution $v_1 \in \mathscr{D}'(0, \infty)$ of the equation $S_1 * v = -S_2 * E_1$. Since S_2 is in

$\mathscr{D}'(\alpha, \infty)$, $S_2 * E_1$ is also in $\mathscr{D}'(\alpha, \infty)$ by (3). But $0 \in \operatorname{sing\,supp} S_1$, and so $v_1 \in \mathscr{D}'(\alpha, \infty)$ by Titchmarsh's theorem. We also have

$$S * (E_1 + v_1) = \delta + S_2 * v_1.$$

Suppose that v_k is defined for $k = 1, 2, \ldots, n$, so that $v_k \in \mathscr{D}'(k\alpha, \infty)$ and

(19)
$$S * \left(E_1 + \sum_{k=1}^{n} v_k \right) = \delta + S_2 * v_n.$$

We define v_{n+1} to be a solution in $\mathscr{D}'(0, \infty)$ of the equation $S_1 * v = -S_2 * v_n$. By the same argument as above, we can show that $v_{n+1} \in \mathscr{D}'((n+1)\alpha, \infty)$ and

$$S * \left(E_1 + \sum_{k=1}^{n+1} v_k \right) = \delta + S_2 * v_{n+1}.$$

We have thus constructed a sequence of distributions $v_n \in \mathscr{D}'(n\alpha, \infty)$ which satisfy (19). But $v_n \to 0$, and therefore $S_2 * v_n \to 0$ as $n \to \infty$, and the series $\sum_{n=1}^{\infty} v_n$ converges in $\mathscr{D}'(0, \infty)$. Consequently, it follows from (19) that $E = E_1 + \sum_{n=1}^{\infty} v_n$ is a fundamental solution for S in $\mathscr{D}'(0, \infty)$.

COROLLARY 2. *If $S \in \mathscr{D}'(0, \infty)$, the following conditions are equivalent.*

(k_1) $S * \mathscr{D}'(0, \infty) = \mathscr{D}'(0, \infty)$.

(k_2) *For some $\alpha > 0$ (or equivalently, for every $\alpha > 0$), S admits a decomposition (14), where $S_1 \in \mathscr{E}'$, $S_2 \in \mathscr{D}'(\alpha, \infty)$, and S_1 satisfies the equivalent conditions in Corollary 1.*

We now establish necessary and sufficient conditions on a distribution $S \in \mathscr{S}'(0, \infty)$ in order that $S * \mathscr{S}'(0, \infty) = \mathscr{S}'(0, \infty)$.

THEOREM 5. *Let $S \in \mathscr{S}'(0, \infty)$ and let $\hat{s}$ be the analytic continuation in C_- of the Fourier transform $\hat{S}$ of S. Then the following conditions are equivalent.*

(l_1) $S * \mathscr{S}'(0, \infty) = \mathscr{S}'(0, \infty)$.

(l_2) $\hat{s}(\zeta) \neq 0$ *for all $\zeta \in C_-$, and there exist positive integers m, n and a constant $c > 0$ such that*

(20)
$$|\hat{s}(\zeta)| \geqslant c(1 + |\zeta|)^{-m} M_{-n}(\eta),$$

where $\zeta = \xi + i\eta$ and M_n is the function defined by (4).

PROOF. It follows from (l_1) that there is $E \in \mathscr{S}'(0, \infty)$ such that

(21)
$$S * E = \delta.$$

If $\hat{e}$ is the analytic continuation in C_- of the Fourier transform $\hat{E}$ of E, then (21) implies that $\hat{s}(\zeta)\hat{e}(\zeta) = 1$, $\zeta \in C_-$. Hence $\hat{s}(\zeta) \neq 0$ for all $\zeta \in C_-$, and there are positive integers m, n and a constant C such that

$$|1/\hat{s}(\zeta)| = |\hat{e}(\zeta)| \leqslant C(1 + |\zeta|)^{m} M_n(\eta),$$

in view of (5). This is equivalent to (20) with $c = 1/C$.

Conversely, if the analytic continuation $\hat{s}$ of $\hat{S}$ satisfies conditions (l_2), then $\hat{e}(\zeta) = 1/\hat{s}(\zeta)$ satisfies the estimate (5) in C_-, and so $\hat{e}(\zeta)$ is the analytic continuation of the Fourier transform $\hat{E}$ of a distribution $E \in \mathscr{S}'(0, \infty)$. Moreover, since $\hat{s}(\zeta)\hat{e}(\zeta) = 1$, we have $S * E = \delta$, i.e. E is a fundamental solution in $\mathscr{S}'(0, \infty)$ for S.

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